LÖWENHEIM–SKOLEM PROBLEM FOR FUNCTORS

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ABSTRACT

An expressibility result is proved for infinitary languages.

§1. Introduction

Let $\mathscr{R}$ be a class of algebraic systems of a fixed similarity type and $\mathscr{H}$ be a class of functors $\mathscr{R} \to$ a class of algebraic systems of possibly another similarity type. Does there exist a cardinal number ה such that $\forall P \in \mathscr{R}$: $\exists Q \in \mathscr{R}$: $Q \subseteq P$ & $|Q| <$ ה & $(\forall H \in \mathscr{H}$: $H(Q) \to H(P)$ is an embedding of an elementary submodel)? I'll prove below an expressibility result for infinitary languages which implies that the answer is positive whenever $\mathscr{R}$ satisfies a kind of Löwenheim–Skolem theorem and elements of $\mathscr{H}$ preserve $\varinjlim$'s.

I met a problem of this kind in topology: find ℸ = min{ה | $\forall$ ring R: $\exists$ subring $R' \subseteq R$: $|R'| <$ ה & $\forall$ compact pair $X \supseteq Y$: $\check{H}(X, Y; R') \to \check{H}(X, Y; R)$ is an embedding of an elementary submodel}, where $\check{H}$ is the functor of Čech cohomologies. I thank the referee for his suggestion to state the main result in a general form mentioning no topological space but only categories and functors. Topological applications are presented at the end of the paper.

§2. Preliminaries

A functor H will be called continuous if $H(\lim -) = \varinjlim H(-)$. A functor will be called algebraic if its values are algebraic systems and homomorphisms.

Let $\mathscr{K}$ be a category and $\mathscr{B}$ be a small subcategory of $\mathscr{K}$. $\mathscr{B}$ will be called a base for $\mathscr{K}$ if every object from $\mathscr{K}$ is $\varinjlim$ of a direct spectrum consisting of objects and arrows from $\mathscr{B}$ and for any $A \to K$, $B \to K$, $A, B \in \mathscr{B}$, $K \in \mathscr{K}$, there exist $A \to C$, $B \to C \in \mathrm{Arr}(\mathscr{B})$, $C \to K$ such that the diagram

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$$\begin{array}{ccccc} A & \longrightarrow & C & \longleftarrow & B \\ & \searrow & \downarrow & \swarrow & \\ & & K & & \end{array}$$

is commutative.

Let $\mathscr{K}$ be a class of algebraic systems of a similarity type σ. $\mathscr{K}$ will be called $\mathfrak{l}$-based ($\mathfrak{l} \geqq \aleph_0 + |\sigma|$) if $\mathscr{K}$ possesses a base $\mathscr{B}$ of cardinality $\leqq \mathfrak{l}$ and such that every object from $\mathscr{B}$ is freely generated in $\mathscr{K}$ by $< \mathfrak{l}$ generators and relations. (Note that every object X is freely generated by $\leqq \aleph_0 + |X| + |\sigma|$ generators and relations.)

Examples. Čech cohomology functor: (the dual to the category of compact pairs) $\times$ rings $\rightarrow$ algebras is continuous and algebraic. Every quasivariety (see [2]) of a similarity type σ is $|\sigma|$-based (the base should consist of finitely presentable objects and their homomorphisms). The class of fields is $\aleph_0$-based.

Expressions like $L_{\lambda\mu}$-elementary, $L_{\lambda\mu}$-sentences, ... will be abbreviated below to $\lambda\mu$-elementary, $\lambda\mu$-sentences, ... $\mathfrak{d}_\xi$'s are defined for any cardinal $\mathfrak{d}$ as follows: $\mathfrak{d}_0 = \mathfrak{d}$, $\mathfrak{d}_{\mu+1} = 2^{\mathfrak{d}_\mu}$, $\mathfrak{d}_\lambda = \sup\{\mathfrak{d}_\mu \mid \mu < \lambda\}$ for limit λ.

§3. Let $\mathscr{K}$ be $\mathfrak{l}$-based, $\mathscr{B}$ be an appropriate base for $\mathscr{K}$ and $\mathscr{H}$ be the class of all continuous algebraic functors defined on $\mathscr{K}$.

For $K \in \mathscr{K}$, $H \in \mathscr{H}$ and $c_1, \cdots, c_n \in H(K)$ one can consider the properties of $\vec{c}$ with respect to $H(K)$ in the following way: find a B in $\mathrm{Ob}(\mathscr{B})$, a homomorphism $k : B \rightarrow K$ and $b_1, \cdots, b_n \in H(B)$ such that $\vec{c}$ is the image of $\vec{b}$ under $H(k) : H(B) \rightarrow H(K)$, and translate sentences about $\vec{c}$ into sentences about images in K of generators of B.

For brevity and simplicity I'll consider only $\infty\omega$-sentences about elements of $H(K)$'s instead of arbitrary $\infty\alpha$-sentences. Define function $\underline{\mathrm{ord}}$ on $\infty\omega$-formulae with finitely many free variables: $\underline{\mathrm{ord}}(\varphi) = \mathfrak{l}$ for atomic φ, $\underline{\mathrm{ord}}(\neg\varphi) = \underline{\mathrm{ord}}(\varphi)$, $\underline{\mathrm{ord}}((\forall x)\varphi) = 2^{\underline{\mathrm{ord}}(\varphi)}$, $\underline{\mathrm{ord}}(\&\{\varphi_t \mid t \in T\}) = \mathfrak{v} + \min(|T|, 2^{\mathfrak{v}})$, where $\mathfrak{v} = \sup\{\underline{\mathrm{ord}}(\varphi_t) \mid t \in T\}$.

Theorem 1. *For any $B \in \mathrm{Ob}(\mathscr{B})$ and a system of its generators of cardinality $< \mathfrak{l}$, $H \in \mathscr{H}$, $b_1, \cdots, b_n \in H(B)$ and $\varphi(x_1, \cdots, x_n) \in L_{\infty\omega}(\tau)$, where τ is the similarity type of values of H, there exists a $\Phi \in L_{\mathfrak{n},\mathfrak{l}}(\sigma)$, where $\mathfrak{n} = (\underline{\mathrm{ord}}(\varphi))^+$, such that for any $K \in \mathscr{K}$ and homomorphism $k : B \rightarrow K$ we have: $\varphi(H(k)(\vec{b}))$ in $H(K)$ if and only if Φ(image in K of a system of generators of B).*

Such Φ will be denoted by $\Phi_{\varphi,B,\vec{b},H}$.

PROOF. Proceed by induction on φ.

(i) Let φ be atomic and Σ be the list of homomorphisms $j: B \rightarrow C \in \mathrm{Arr}(\mathscr{B})$ such that $\varphi(H(j)(\vec{b}))$. Augment each such j with a system of cardinality $< \text{ו}$ of generators and defining relations for C, $\Xi = \{(j, \text{generators and relations}) \mid j \in \Sigma\}$. Put $\Phi(\vec{g}) = \bigvee\{S_\xi(\vec{g}) \mid \xi \in \Xi\}$, where $S_\xi(\vec{g}) = \exists \vec{g}_1$: $\langle\langle \vec{g}_1$ satisfies defining relations of ξ for generators of C and $\vec{g}$ is mapped into C according to $\xi \rangle\rangle$, where $\vec{g}_1$ is of appropriate cardinality. It is clear that $\langle\langle \quad \rangle\rangle$ above is expressible in $L_{\gimel}(\sigma)$ and since $|\Xi| \leqq \text{ו}$ we have $\Phi \in L_{\beth,\text{ו}}(\sigma)$.

(ii) Put $\Phi_{\neg\varphi,\cdots} = \neg\Phi_{\varphi,\cdots}$ and $\Phi_{\&_\iota \varphi_\iota \cdots} = \&_\iota \Phi_{\varphi_\iota \cdots}$.

(iii) Let $\varphi = (\forall x)\psi(x, \cdots)$. Let Ξ be the list of all homomorphisms $B \rightarrow C \in \mathrm{Arr}(\mathscr{B})$ each augmented with a system of cardinality $< \text{ו}$ of generators and defining relations for C. Put $\Phi_{\varphi,B,\vec{b},H} = \&\{S_\xi(\vec{g}) \mid \xi \in \Xi\}$, where $S_\xi(\vec{g}) = \&\{(\forall \vec{g}_1)(\langle\langle \vec{g}_1$ satisfies defining relations of ξ for generators of C and $\vec{g}$ is mapped into C according to $\xi\rangle\rangle \Rightarrow \Phi_{\psi,C,\vec{c}\cdot\varsigma,H}(\vec{g}_1)) \mid \varsigma \in H(C)\}$, where $\vec{g}_1$ is of appropriate cardinality and $\vec{c}$ is the image of $\vec{b}$ under the homomorphism $H(B) \rightarrow H(C)$ corresponding to ξ. Since $\Phi_{\psi,\cdots}$'s belong to $L_{\gimel}(\sigma)$, where $\gimel = (\underline{\mathrm{ord}}(\psi))^+$, we have $\Phi_{\varphi,\cdots} \in L_{\beth,\text{ו}}(\sigma)$.

COROLLARY 2. *Let* $A, B \in \mathscr{K}$, $H \in \mathscr{H}$ *and* $\daleth > \text{ו}$. *If* A, B *are* $\daleth,\text{ו}$*-equivalent then* $H(A)$, $H(B)$ *are equivalent with respect to sentences of* $\underline{\mathrm{ord}} < \daleth$. *If* $A \subseteq B$ *and* A *is* $\daleth,\text{ו}$*-elementary in* B *then the corresponding* $H(A) \rightarrow H(B)$ *is an embedding and the image of* $H(A)$ *is elementary in* $H(B)$ *with respect to formulae of* $\underline{\mathrm{ord}} < \daleth$.

$\mathscr{K}$ *will be called* כ*-Löwenheim if for any* $A \in \mathscr{K}$ *and a set* $X \subseteq A$ *there exists* $B \in \mathscr{K}$ *such that* $X \subseteq B \subseteq A$ *and* $|B| \leqq |X| + \text{כ}$.

COROLLARY 3. *If* $\mathscr{K}$ *is* כ*-Löwenheim and is closed under unions of increasing sequences, then for any* $A \in \mathscr{K}$ *there exists* $B \in \mathscr{K}$ *such that* $B \subseteq A$, $H(B) \rightarrow H(A)$ *for any* $H \in \mathscr{H}$ *is an embedding of an elementary submodel, and* $|B| \leqq \gimel_\omega$ *with* $\gimel = \text{כ}$.

§4. Topological remarks

COROLLARY 4. *For* ר *mentioned in the introduction*: $\text{ר} \leqq (\gimel_\omega)^+$ *with* $\gimel = \aleph_0$.

Let H be an algebraic functor continuous on the dual of the category of compact pairs (or other combinations of compact spaces likely for topology). Since $H(X) = \varinjlim H$(the spectrum of all PL-approximations of X) for any X, the class of all continuous H's with values of a fixed similarity type τ can be regarded as a single continuous functor on (the dual of the category of compact pairs) $\times$ (the class of algebraic systems of the similarity type $\tau^\wedge$ satisfying appropriate equalities), where $\tau^\wedge = \tau \cup$ (relations corresponding to values of H on PL-pairs, and operations corresponding to PL-maps).

COROLLARY 5. *For any contravariant continuous algebraic functor H on the category of compact pairs there exist a functor H_1 of the same kind and transformation $H_1 \to H$ such that $H_1(X) \to H(X)$ is an embedding of an elementary submodel for any X and $|H_1(X)| \leqq \text{ט} + \gimel_\omega$ for $|X| \leqq \text{ט} + \gimel_\omega$, with $\gimel = \aleph_0$ and $\text{ט} = |$similarity type of values of $H|$.*

I don't know whether the above cardinal bounds are exact. Doubts are based in part on the following result (proof will appear elsewhere).

PROPOSITION 6. *For any compact pair $X \supseteq Y$ and a ring R there exist a countable subring $R' \subseteq R$, metric compacts $X' \supseteq Y'$ and an onto map $(X, Y) \to (X', Y')$ such that $\check{H}(X', Y'; R') \to \check{H}(X, Y; R)$ is an embedding of an elementary submodel.*

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